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NEW JERSEY CENTER
FOR TEACHING & LEARNING

AP Calculus AB

Limits & Continuity

2015-10-20

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Introduction

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The History of Calculus

Calculus is the Latin word for stone. In Ancient times, the Romans used stones for counting and basic arithmetic. Today, we know Calculus to be very special form of counting. It can be used for solving complex problems that regular mathematics cannot complete. It is because of this that Calculus is the next step towards higher mathematics following Advanced Algebra and Geometry.

In the 21st century, there are so many areas that required Calculus applications: Economics, Astronomy, Military, Air Traffic Control, Radar, Engineering,Medicine, etc.

The History of Calculus

The foundation for the general ideas of Calculus come from ancient times but Calculus itself was invented during the 17th century. The first principles were presented by **Sir Isaac Newton** of England, and the German mathematician **Gottfried Wilhelm Leibnitz.**

The History of Calculus

Both Newton and Leibnitz deserve equal credit for independently coming up with calculus. Historically, each accused the other for plagiarism of their Calculus concepts but ultimately their separate but combined works developed our first understandings of Calculus.

Newton was also able to establish our first insight into physics which would remain uncontested until the year 1900. His first works are still in use today.

The History of Calculus

The two main concepts in the study of Calculus are differentiation and integration. Everything else will concern ideas, rules, and examples that deal with these two principle concepts.

Therefore, we can look at Calculus has having two major branches: Differential Calculus (the rate of change and slope of curves) and Integral Calculus (dealing with accumulation of quantities and the areas under curves).

The History of Calculus

Calculus was developed out of a need to understand continuously changing quantities.

Newton, for example, was trying to understand the effect of gravity which causes falling objects to constantly accelerate. In other words, the speed of an object increases constantly as it falls. From that notion, how can one say determine the speed of a falling object at a specific instant in time (such as its speed as it strikes the ground)? No mathematicians prior to Newton / Leibnitz's time could answer such a question. It appeared to require the impossible: dividing zero by zero.

The History of Calculus

Differential Calculus is concerned with the continuous / varying change of a function and the different applications associated with that function. By understanding these concepts, we will have a better understanding of the behavior(s) of mathematical functions.

Importantly, this allows us to optimize functions. Thus, we can find their maximum or minimum values, as well as **#** determine other valuable qualities that can describe the function. The real-world applications are endless: maximizing profit, minimizing cost, maximizing efficiency, finding the point of diminishing returns, determining velocity/acceleration, etc.

The History of Calculus

The other branch of Calculus is **Integral Calculus.** Integration is the process which is the reverse of differentiation. Essentially, it allows us to add an infinite amount of infinitely small numbers. Therefore, in theory, we can find the area / volume of any planar geometric shape. The applications of integration, like differentiation, are also quite extensive.

The History of Calculus

These two main concepts of Calculus can be illustrated by real-life examples:

1) <u>"How fast is a my speed changing with time?"</u>
For instance, say you're driving down the highway:
Let *s* represents the distance you've traveled. You might be interested in how fast *s* is changing with time. This quantity is called velocity, v.
Studying the rates of change involves using the derivative. Velocity is the derivative of the position function s. If we think of our distance *s* as a function of time denoted s = f(t), then we can express the derivative $v = ds/dt$. (change in distance over change in time)

The History of Calculus

Whether a rate of change occurs in biology, physics, or economics, the same mathematical concept, the derivative, is involved in each case.

The History of Calculus

2) "How much has a quantity changed at a given time?"
This is the "opposite" of the first question. If you know how fast a quantity is changing, then do you how much of an impact that change has had?

On the highway again: You can imagine trying to figure out how far, s, you are at any time *t* by studying the velocity *v*.

This is easy to do if the car moves at constant velocity:

In that case, distance = (velocity)(time), denoted $s = v*t$.

But if the car's velocity varies during the trip, finding s is a bit harder. We have to calculate the total distance from the function $v = ds/dt$. This involves the concept of the integral.

1 What is the meaning of the word Calculus in Latin?

- A Count
- B Stone
- C Multiplication
- D Division
- E None of above

1 What is the meaning of the word Calculus in Latin?

- A Count
- B Stone
- C Multiplication
- D Division
- E None of above

Answer

B

2 Who would we consider as the founder of Calculus?

- A Newton
- B Einstein
- C Leibnitz
- D Both Newton and Einstein
- E Both Newton and Leibnitz

2 Who would we consider as the founder of Calculus?

- A Newton
- B Einstein
- C Leibnitz
- D Both Newton and Einstein
- E Both Newton and Leibnitz

Answer

E

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3 What areas of life do we use calculus?

- A Engineering
- B Physical Science
- C Medicine
- D Statistics
- E Economics
- F Chemistry
- G Computer Science
- H Biology
- I Astronomy
- J All of above

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3 What areas of life do we use ca

- A Engineering
- B Physical Science
- C Medicine
- D Statistics
- E Economics
- F C
- G Co
- H Bio
- I Astronomy
- J All of above

Answer

J

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4 How many major concepts does the study of Calculus have

- A Three
- B Two
- C One
- D None of above

4 How many major concepts does have

- A Three
- B Two
- C One
- D None of above

Answer

B

5 What are the names for the main branches of Calculus?

- A Differential Calculus
- B Integral Calculus
- C Both of them

5 What are the names for the ma us?

- A Differential Calculus
- B Integral Calculus
- C Both of them

Answer

C

The History of Calculus

The preceding information makes it clear that all ideas of Calculus originated with the following two geometric problems:

1. The Tangent Line Problem
Given a function *f* and a point *P(x0, y0)* on its graph, find an equation of the line that is tangent to the graph at *P*.

2. The Area Problem
Given a function *f*, find the area between the graph of *f* and an interval [*a*,*b*] on the x-axis.

In the next section, we will discuss The Tangent Line problem.

This will lead us to the definition of the limit and eventually to the definition of the derivative.

The Tangent Line Problem

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The Tangent Line Problem

In plane geometry, the tangent line at a given point (known simply as the tangent) is defined as the straight line that meets a curve at precisely one point (Figure 1). However, this definition is not appropriate for all curves. For example, in Figure 2, the line meets the curve exactly once, but it obviously not a tangent line. Lastly, in Figure 3, the tangent line happens to intersect the curve more than once.

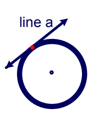

Figure 1.

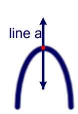

Figure 2.

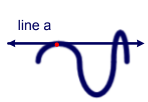

Figure 3.

The Tangent Line Problem

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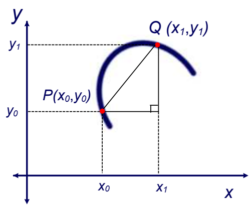
Let us now discuss a problem that will help to define a slope of a tangent line. Suppose we have two points, $P(x_0,y_0)$ and $Q(x_1,y_1)$, on the curve. The line that connects those two points is called the secant line (called the secant).
We now find the slope of the secant line using very familiar algebra formulas:

$$m_{\text{sec}} = \frac{\text{rise}}{\text{run}} = \frac{y_1 - y_0}{x_1 - x_0}$$



The Tangent Line Problem

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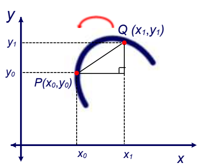
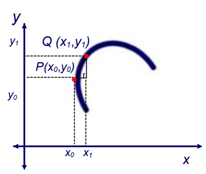
If we move the point Q along the curve towards point P, the distance between x_1 *and* x_0 *gets smaller and smaller* and the difference x_1-x_0 *will approach zero.*





The Tangent Line Problem

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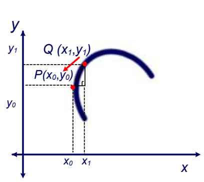
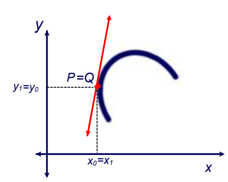
Eventually points *P* and Q will coincide and the secant line will be in its *limiting* position. Since P and Q are now the same point, we can consider it to be a tangent line.





The Tangent Line

Now we can state a precise definition.
A Tangent Line is a secant line in its limiting position.
The slope of the tangent line is defined by following formula:

$$m_{tan} = m_{sec} = \frac{y_1 - y_0}{x_1 - x_0}$$, when x_1 approaches to x_0 ($x_1 \to x_0$), so $x_1 = x_0$.

Formula 1.

The Tangent Line

The changes in the x and y coordinates are called increments.

As the value of x changes from x_1 to x_2, then we denote the change in x as $\Delta x = x_2 - x_1$. This is called the increment within x.

The corresponding changes in y as it goes from y_1 to y_2 are denoted $\Delta y = y_2 - y_1$. This is called the increment within y.

Then Formula 1 can be written as:

$$m_{tan} = \frac{\Delta y}{\Delta x}$$, when x_2 approaches x_1, ($x_2 \to x_1$), so $\Delta x \to 0$.

Formula 1a.

The Tangent Line

Note: We can also label our y's as $y_1 = f(x_1)$ and $y_2 = f(x_2)$.
Therefore, we can say that

$$m_{sec} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$, *which will imply*

$$m_{tan} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta f(x)}{\Delta x}$$, *when* $\Delta x \to 0$.

Formula 1b.

The Formula 1b is just another definition for the slope of the tangent line.

The Tangent Line

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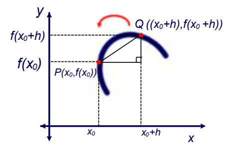
Now we can use a familiar diagram, with the new notation to represent an alternative formula for the slope of a tangent line.



Note.
When point Q moves along the curve toward point P, we can see that $h \to 0$.

$$m_{tan} = \frac{f(x_0+h) - f(x_0)}{x_0+h - x_0} = \frac{f(x_0+h) - f(x_0)}{h}, \text{ when } h \to 0.$$

Formula 1c.

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6 What is the coordinate increment in x from A(-2, 4) to B(2,-3)?

- A -4
- B 7
- C -7
- D 0
- E 4

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6 What is the coordinate increment in x from A(-2, 4) to B(2,-3)?

- A -4
- B 7
- C -7
- D 0
- E 4

Answer

E

7 What is the coordinate increment in y from A(-2,4) to B(2,-3)?

- A -4
- B 7
- C -7
- D 0
- E 4

7 What is the coordinate increm
to B(2,-3)?

- A -4
- B 7
- C -7
- D 0
- E 4

Answer

C

Example 1

For the function $f(x) = x^2 - 1$, find the following:

a. the slope of the secant line between $x_1 = 1$ and $x_2 = 3$;
b. the slope of the tangent line at $x_0 = 2$;
c. the equation of the tangent line at $x_0 = 2$.

a. the slope of the secant line between $x_1 = 1$ and $x_2 = 3$;

Let us use one of the formulas for the secant lines:

$$m_{\sec} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} =$$

Example 1

For the function *f (x) = x²-1*, f

a. the slope of the secant line
b. the slope of the tangent li
c. the equation of the tange

Answer

$$m_{\sec} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(3) - f(1)}{3 - 1} =$$

$$\frac{(3^2 - 1) - (1^1 - 1)}{2} = \frac{8}{2} = 4$$

a. the slope of the secant line

Let us use one of the formula

$$m_{\sec} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} =$$

Example 2

For the function $f(x) = 3x+11$, find the following:

a. the slope of the secant line between $x_1 = 2$ and $x_2 = 5$;
b. the slope of the tangent line at $x_0 = 3$;
c. the equation of the tangent line at $x_0 = 3$.

Example 2

For the fun

a. the slo ... $x_2 = 5$;
b. the slo
c. the eq

Answer

a. 3;
b. 3;
c. y=3x+11

*Note, that this result is obvious:
the given equation represents a line.
Line has the same slope at all points,
and it's a tangent to itself.

Example 3

For the function $f(x) = 2x^2-3x+1$, find the following:

a. the slope of the secant line between $x_1 = 1$ and $x_2 = 3$;
b. the slope of the tangent line at $x_0 = 2$;
c. the equation of the tangent line at $x_0 = 2$.

Example 3

For the function *f (x*

a. the slope of the s
b. the slope of the t
c. the equation of th

Answer

a. 5;
b. 5;
c. y=5x-7

*Note: At some points on interval the secant line may have the same slope as a tangent line. We will discuss this result again later on in the course.

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Example 4

For the function $f(x) = x^3+2x^2-1$, find the following:

a. the slope of the secant line between $x_1 = 2$ and $x_2 = 4$;
b. the slope of the tangent line at $x_0 = 3$;
c. the equation of the tangent line at $x_0 = 3$.
Use formula 1b for part b.

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Example 4

For the function $f(x) = x^3+2x^2-1$, find the following:

a. the slope of the secant line between $x_1 = 2$ and $x_2 = 4$;
b. the slope of the tangent line at $x_0 = 3$;
c. the equation of the tangent line at $x_0 = 3$.
Use formula 1b for part b.

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Example 5

For the function $f(x) = \frac{2}{x}$, find the following:

a. the slope of the secant line between $x_1 = 3$ and $x_2 = 6$;
b. the slope of the tangent line at $x_0 = 4$;
c. the equation of the tangent line at $x_0 = 4$.
Use formula 1c for part b.

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Definition of a Limit and Graphical Approach

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0 in the Denominator

In the previous section, when we were trying to find a general formula for the slope of a tangent line, we faced a certain difficulty:

The denominator of the fractions that represented the slope of the tangent line always went to zero.

You may have noticed that we avoided saying that the denominator equals zero. With Calculus, we will use the expression "approaching zero" for these cases.

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Limits

There is an old phrase that says to "Reach your limits": Generally it's used when somebody is trying to reach for the best possible result. You will also implicitly use it when you slow down your car when you can see the speed limit sign.

You may even recall from the previous section that when one point is approaching another, the secant line becomes a tangent
line in what we consider to be the *limiting* position of a secant line.

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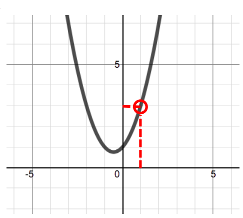
Limits

Now we will discuss a certain algebra problem.

Suppose you want to graph a function: $f(x)=\dfrac{x^3-1}{x-1}$

For all values of x, except for x = 1, you can use standard curve sketching techniques. The reason it has no value for x = 1 is because the curve is not defined there. This is called an unknown, or a "hole" in the graph.



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Limits

In order to get an idea of the behavior of the curve around $x = 1$ we will complete the chart below:

x	0.75	0.95	0.99	0.999	1.00	1.001	1.01	1.1	1.25
f(x)	2.3125	2.8525	2.9701	2.9970		3.003	3.030	3.310	3.813

You can see that as x gets closer and closer to 1, the value of $f(x)$ comes closer and closer to 3. We will say that the ***limit*** of $f(x)$ as x approaches 1, is 3 and this is written as

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

Limits

The informal definition of a limit is:

"What is happening to y as x gets close to a certain number."

The function doesn't have to have an actual value at a particular x for the limit to exist. Limits describe what happens to a function as x *approaches* the value. In other words, a limit is the number that the value of a function "should" be equal to and therefore is trying to reach.

Formal Definition of a Limit

We say that the limit of $f(x)$ is L as x approaches c provided that we can make f(x) as close to L as we want for all x sufficiently close to c, from both sides, without actually letting x be c.
This is written as $\lim_{x \to c} f(x) = L$

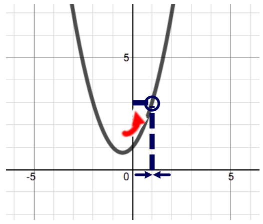
and it is read as "The limit of f of x, as x approaches c, is L.
As we approach c from both sides, sometimes we call this type of a limit ***a two-sided limit***.

Two-Sided Limit

In our previous example, as *we approach* 1 from the left (it means that value of x is slightly smaller than 1), the value of *f*(*x*) becomes closer and closer to 3.

As *we approach* 1 from the right (it means that value of x is slightly greater than 1), the value of f(x) is also getting closer and closer to 3.

The idea of approaching a certain number on x-axis from different sides leads us to the general idea of ***a two-sided limit*** .



Left and Right Hand Limits

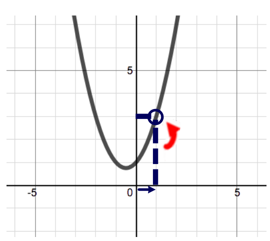
If we want the limit of *f* (*x*) as we approach the value of *c* from the left hand side, we will write $\lim_{x\to c^-} f(x)$.

If we want the limit of *f (x)* as we approach the value of c from the right hand side, we will write $\lim_{x\to c^+} f(x)$.

Left Hand Limit

The one-sided limit of *f* (*x*) as *x* approaches 1 from the left will be written as

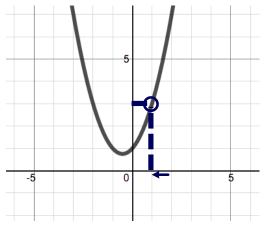
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^3 - 1}{x - 1} = 3 .$$



Right Hand Limit

The one-sided limit of *f* (*x*) as *x* approaches 1 from the right will be written as

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x^3 - 1}{x - 1} = 3 .$$



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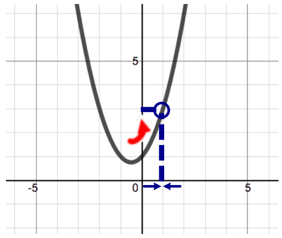
LHL=RHL

So, in our example

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = \lim_{x\to -1} f(x) = 3$

Notice that f(c) doesn't have to exist, just that coming from the right and coming from the left the function needs to be going to the same value.



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Limits with Graphs - Example 1

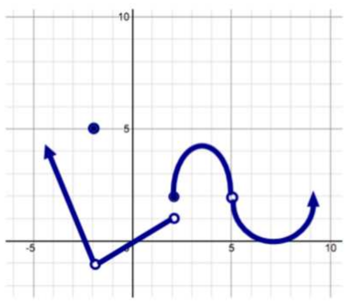
Use the graph to find the indicated limit.

$\lim_{x\to -2^-} f(x) = \_\_\_\_\_$

$\lim_{x\to -2^+} f(x) = \_\_\_\_\_$

$\lim_{x\to -2} f(x) = \_\_\_\_\_$

$f(-2) = \_\_\_\_\_$



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Limits with Graphs - Example 1

Use the graph to find the indicated limit.

$\lim_{x\to -2^-} f(x) = \_\_\_\_\_$

$\lim_{x\to -2^+} f(x) = \_\_\_\_\_$

$\lim_{x\to -2} f(x) = \_\_\_\_\_$

$f(-2) = \_\_\_\_\_$

Answer

$\lim_{x\to -2^-} f(x) = -1$

$\lim_{x\to -2^+} f(x) = -1$

$\lim_{x\to -2} f(x) = -1$

$f(-2) = 5$

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Limits with Graphs - Example 2

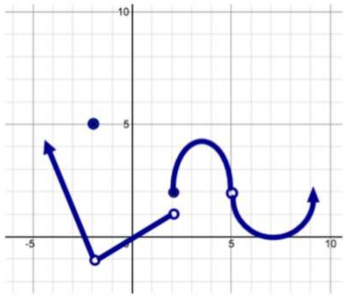
Use graph to find the indicated limit.

$\lim_{x \to 0^-} f(x) =$ ______

$\lim_{x \to 0^+} f(x) =$ ______

$\lim_{x \to 0} f(x) =$ ______

$f(0) =$ ______



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Limits with Gr

Use graph to find the indicated

$\lim_{x \to 0^-} f(x) =$ ______

$\lim_{x \to 0^+} f(x) =$ ______

$\lim_{x \to 0} f(x) =$ ______

$f(0) =$ ______

Answer

$\lim_{x \to 0^-} f(x) = 0$

$\lim_{x \to 0^+} f(x) = 0$

$\lim_{x \to 0} f(x) = 0$

$f(0) = 0$

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Limits with Graphs - Example 3

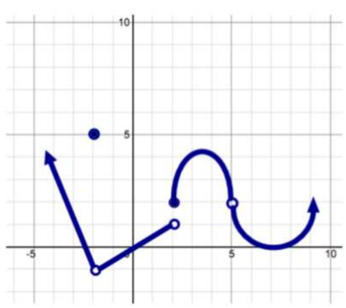
Use graph to find the indicated limit.

$\lim_{x \to 2^-} f(x) =$ ______

$\lim_{x \to 2^+} f(x) =$ ______

$\lim_{x \to 2} f(x) =$ ______

$f(2) =$ ______



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Limits with Graphs - Example 3

Use graph to find the indicate

$\lim_{x\to 2^-} f(x) =$ ______

$\lim_{x\to 2^+} f(x) =$ ______

$\lim_{x\to 2} f(x) =$ ______

$f(2) =$ ______

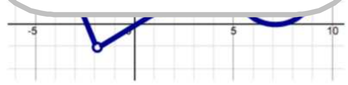
Answer

$\lim_{x\to 2^-} f(x) = 1$

$\lim_{x\to 2^+} f(x) = 2$

$\lim_{x\to 2} f(x) = DNE$

$f(2) = 2$



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Limits with Graphs - Example 4

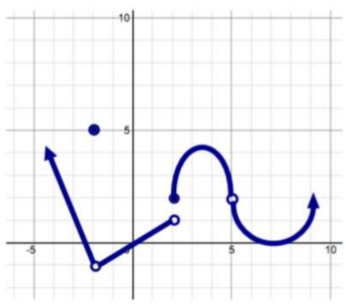
Use graph to find the indicated limit.

$\lim_{x\to 5^-} f(x) =$ ______

$\lim_{x\to 5^+} f(x) =$ ______

$\lim_{x\to 5} f(x) =$ ______

$f(5) =$ ______



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Limits with Graphs - Example 4

Use graph to find the indicated

$\lim_{x\to 5^-} f(x) =$ ______

$\lim_{x\to 5^+} f(x) =$ ______

$\lim_{x\to 5} f(x) =$ ______

$f(5) =$ ______

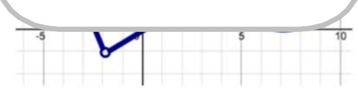
Answer

$\lim_{x\to 5^-} f(x) = 2$

$\lim_{x\to 5^+} f(x) = 2$

$\lim_{x\to 5} f(x) = 2$

$f(5) = DNE$



Slide 55 (Answer) / 233

Limits with Graphs - Example 5

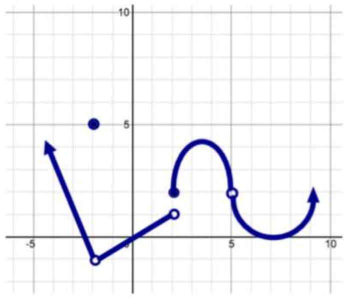
Use graph to find the indicated limit.

$\lim_{x \to 7^-} f(x) =$ ______

$\lim_{x \to 7^+} f(x) =$ ______

$\lim_{x \to 7} f(x) =$ ______

$f(7) =$ ______



Limits with Graphs - Example 5

Use graph to find the indicated

$\lim_{x \to 7^-} f(x) =$ ______

$\lim_{x \to 7^+} f(x) =$ ______

$\lim_{x \to 7} f(x) =$ ______

$f(7) =$ ______

Answer

$\lim_{x \to 7^-} f(x) = 0$

$\lim_{x \to 7^+} f(x) = 0$

$\lim_{x \to 7} f(x) = 0$

$f(7) = 0$

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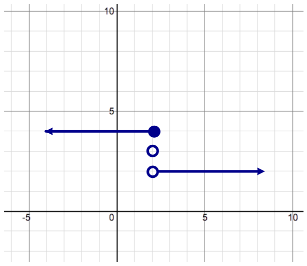
Slide 58 / 233

Slide 58 (Answer) / 233

10 Use the given graph to answer true/false statement:

$f(2) = DNE$



- True
- False

10 Use the given graph t

$f(2) = DNE$

- True
- False

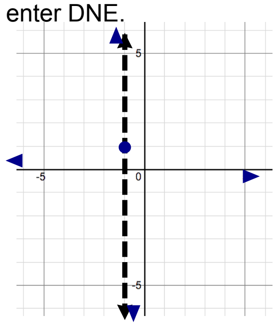
Answer

False
f(2) = 4

Slide 60 (Answer) / 233

12 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter DNE.

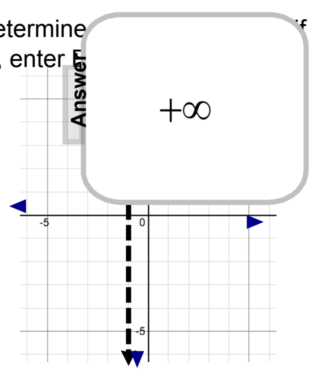
$$\lim_{x \to -1^-} f(x) = \_\_$$



Slide 61 / 233

12 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter DNE.

$$\lim_{x \to -1^-} f(x) = \_\_$$



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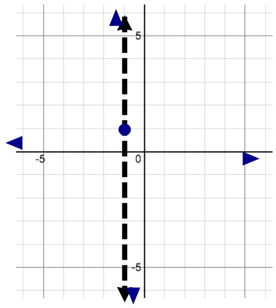
Slide 62 / 233

13 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter DNE.

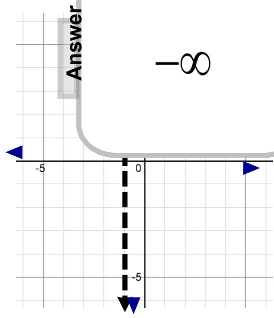
$$\lim_{x\to -1^{+}} f(x) = \_\_$$



Slide 62 (Answer) / 233

13 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter D

$$\lim_{x\to -1^{+}} f(x) = \_\_$$



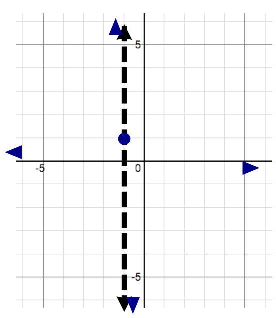
Answer

$-\infty$

Slide 63 / 233

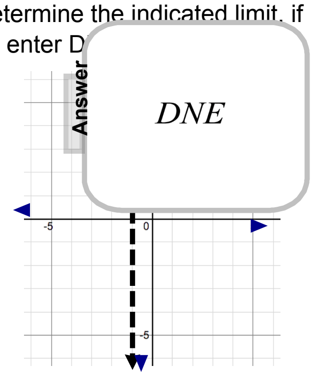
14 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter DNE.

$$\lim_{x\to -1} f(x) = \_\_$$



14 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter D

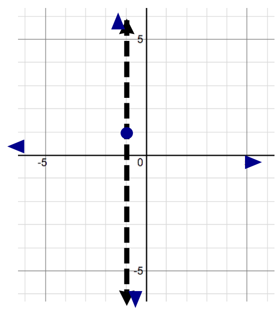
$$\lim_{x \to -1} f(x) = \_\_$$



Slide 63 (Answer) / 233

15 Use the given graph to determine the following value, if it exists. If it doesn't exist, enter DNE.

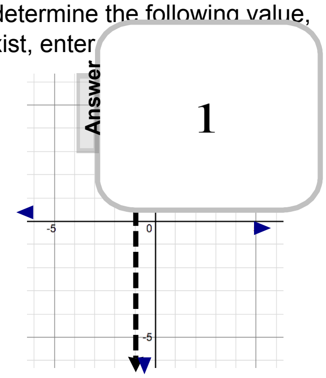
$$f(-1) = \_\_$$



Slide 64 / 233

15 Use the given graph to determine the following value, if it exists. If it doesn't exist, enter

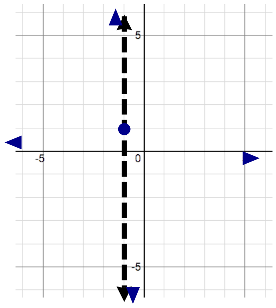
$$f(-1) = \_\_$$



Slide 64 (Answer) / 233

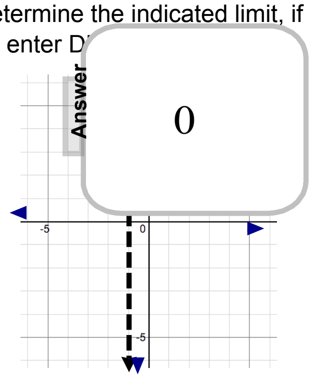
16 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter DNE.

$$\lim_{x \to -\infty} f(x) = \_\_$$



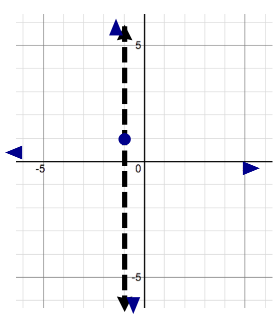
16 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter D

$$\lim_{x \to -\infty} f(x) = \_\_$$



17 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter DNE.

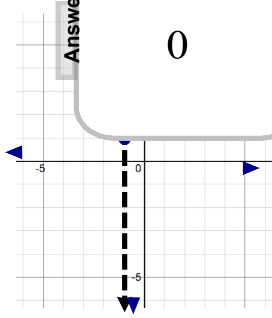
$$\lim_{x \to +\infty} f(x) = \_\_$$



Slide 66 (Answer) / 233

17 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter D

$$\lim_{x\to+\infty} f(x) = \_\_$$



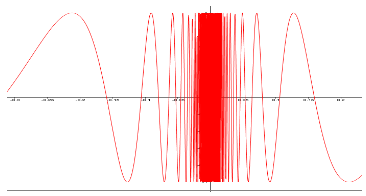
Answer

0

Slide 67 / 233

18 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter DNE.

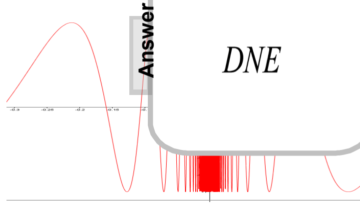
$$\lim_{x\to 0} f(x) = \_\_$$



Slide 67 (Answer) / 233

18 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter D

$$\lim_{x\to 0} f(x) = \_\_$$

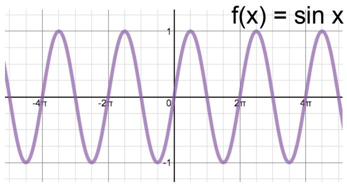


Answer

DNE

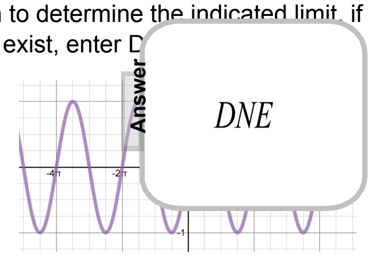
19 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter DNE.

$$\lim_{x \to -\infty} f(x) = \_\_$$



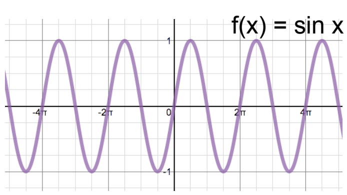
19 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter D

$$\lim_{x \to -\infty} f(x) = \_\_$$



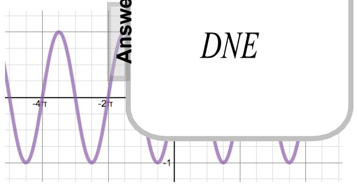
20 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter DNE.

$$\lim_{x \to +\infty} f(x) = \_\_$$



20 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter D

$$\lim_{x \to +\infty} f(x) = \_\_$$

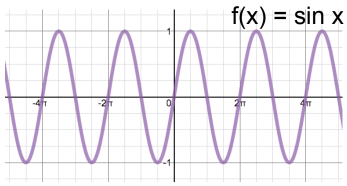


Answer

DNE

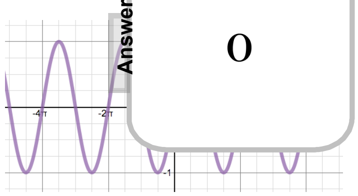
21 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter DNE.

$$\lim_{x \to 0} f(x) = \_\_$$



21 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter D

$$\lim_{x \to 0} f(x) = \_\_$$



Answer

0

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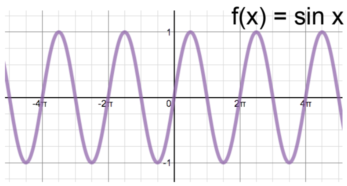
22 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter DNE.

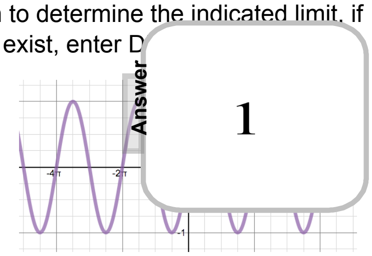
$$\lim_{x\to\frac{\pi}{2}} f(x) = \_\_$$



Slide 71 (Answer) / 233

22 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter D

$$\lim_{x\to\frac{\pi}{2}} f(x) = \_\_$$



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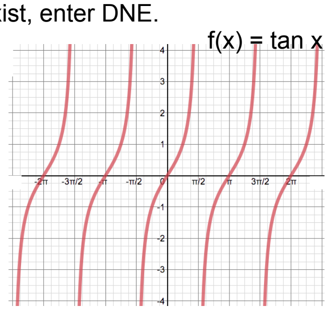
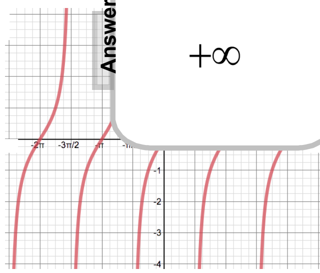
23 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter DNE.

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \_\_$$



23 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter D

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \_\_$$



Answer

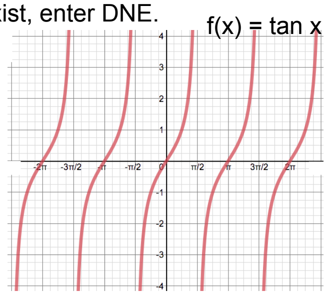
$+\infty$

Slide 72 (Answer) / 233

24 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter DNE.

$$\lim_{x \to \frac{\pi}{2}^+} f(x) = \_\_$$

f(x) = tan x



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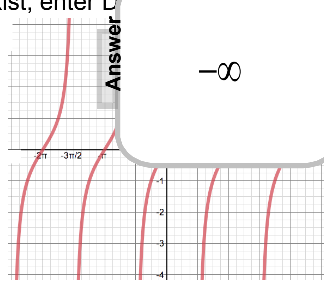
24 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter D

$$\lim_{x \to \frac{\pi}{2}^+} f(x) = \_\_$$



Answer

$-\infty$

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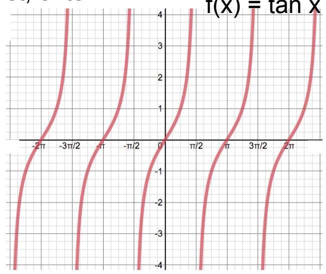
25 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter DNE.

$$\lim_{x \to \frac{\pi}{2}} f(x) = \_\_$$

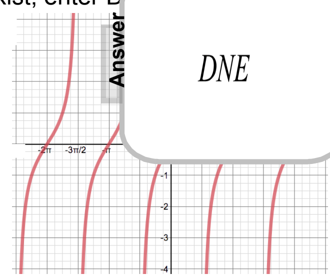
f(x) = tan x



Slide 74 (Answer) / 233

25 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter D

$$\lim_{x \to \frac{\pi}{2}} f(x) = \_\_$$



Answer

DNE

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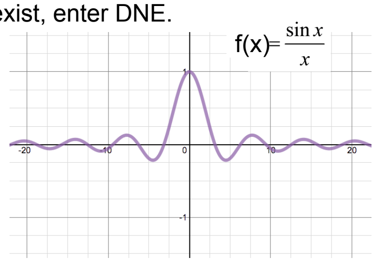
26 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter DNE.

$$\lim_{x \to +\infty} f(x) = \_\_$$

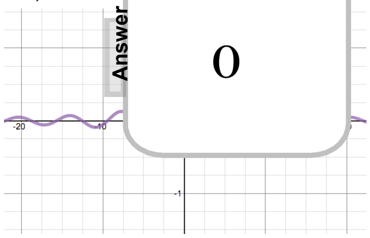
f(x)= $\frac{\sin x}{x}$



Slide 75 (Answer) / 233

26 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter D

$$\lim_{x \to +\infty} f(x) = \_\_$$



Answer

0

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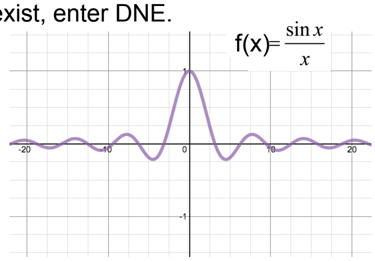
27 Use the given graph to determine the indicated limit, if it exists. If it doesn't exist, enter DNE.

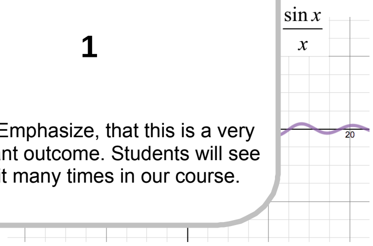
$$\lim_{x \to 0} f(x) = \_\_$$



Slide 76 (Answer) / 233

27 Use the given limit, if it exists. If it

$$\lim_{x \to 0} f(x) =$$



Answer

1

*Note: Emphasize, that this is a very important outcome. Students will see this limit many times in our course.

Computing Limits

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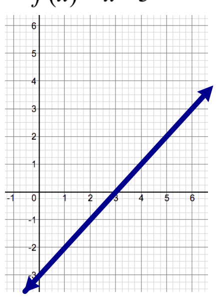
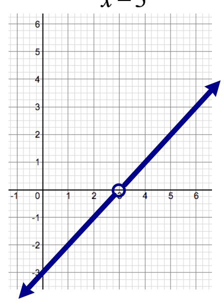
Contents

Let us consider two functions: *f(x)* and *g(x),* as x approaches 3.

$f(x) = x - 3$



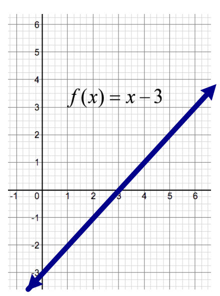
$g(x) = \dfrac{x^2 - 6x + 9}{x - 3}$



Limit Graphically

From the graphical approach
it is obvious that f(x) is a line,
and as x approaches 3 the value
of function *f(x)* will be equal to zero.

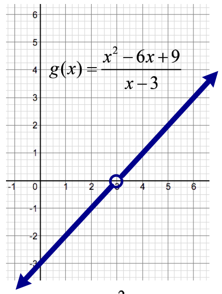
$$\lim_{x \to 3} f(x) = \lim_{x \to 3}(x - 3) = 0$$



Limit Graphically

What happens in our second case?
There is no value of for *g(x)* when x=3. If we remember that a limit describes what happens to a function as it gets closer and closer to a certain value of x, the function doesn't need to have a value at that x, for the limit to exist.

From a graphical point of view, as x gets close to 3 from both the left and right sides, the value of function *g(x)* will approach zero.



$$\lim_{x\to 3} g(x) = \lim_{x\to 3}\frac{x^2-6x+9}{x-3} = 0$$

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Slide 84 (Answer) / 233

Examples:

$1. \lim_{x \to 0} 3x =$ ________

$2. \lim_{x \to 1} 5\sqrt{x-1} =$ ________

$3. \lim_{x \to 3} \frac{5x+1}{2x-1} =$ ________

$4. \lim_{x \to -2} (x^2+5) =$ ________

$5. \lim_{x \to h} \frac{4x^2}{2x} =$ ________

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Examples:

$1. \lim_{x \to 0} 3x =$ ________

$2. \lim_{x \to 1} 5\sqrt{x-1} =$ ________

$3. \lim_{x \to 3} \frac{5x+1}{2x-1} =$ ________

$4. \lim_{x \to -2} (x^2+5) =$ ________

$5. \lim_{x \to h} \frac{4x^2}{2x} =$ ________

Answer

$1. \lim_{x \to 0} 3x = 0$

$2. \lim_{x \to 1} 5\sqrt{x-1} = 0$

$3. \lim_{x \to 3} \frac{5x+1}{2x-1} = \frac{16}{5}$

$4. \lim_{x \to -2} (x^2+5) = 9$

$5. \lim_{x \to h} \frac{4x^2}{2x} = 2h$

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Substitution with One-Sided Limits

You can apply the substitution method for one-sided limits as well. Simply substitute the given number into the expression of a function without paying attention if you are approaching from the right or left.

Approaches 1 from the right only.

$$\lim_{x \to 2^+} \sqrt{x-1} = 1$$

Approaches 1 from the left only.

$$\lim_{x \to 2^-} \sqrt{x-1} = 1$$

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28 Find the indicated limit.

$$\lim_{x \to 7} 2x$$

28 Find the indicated li

$$\lim_{x \to 7} 2x$$

Answer

14

29 Find the indicated limit.

$$\lim_{x \to -8} \frac{x}{x+4}$$

29 Find the indicated li

$$\lim_{x\to -8} \frac{x}{x+4}$$

Answer

2

30 Find the indicated limit.

$$\lim_{x\to 11} \sqrt{2x-6}$$

30 Find the indicated li

$$\lim_{x\to 11} \sqrt{2x}$$

Answer

4

31 Find the indicated limit.

$$\lim_{x\to 2^+} \sqrt{x-2}$$

31 Find the indicated li

$$\lim_{x\to 2^+} \sqrt{x-2}$$

Answer

0

The Indeterminate Form of 0/0

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Zero in Numerator & Denominator

What about our previous problem $g(x) = \dfrac{x^2 - 6x + 9}{x - 3}$?

Substitution will not work in this case. When you plug 3 into the equation, you will get zero on top and zero on bottom. Thinking back to Algebra, when you plug a number into an equation and you got zero, we called that number a root. Now when we get 0/0, that means our numerator and denominator *share* a root. In this case, we then factor the numerator to find that root and reduce. When we solve this problem, we get the predicted answer.

$$\lim_{x \to 3} g(x) = \lim_{x \to 3} \frac{x^2 - 6x + 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x-3)}{x - 3} = \lim_{x \to 3} (x - 3) = 0$$

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Indeterminate Form

A limit where both the numerator and the denominator have the limit zero, as x approaches a certain number, is called a limit with **an indeterminate form 0/0**.

Limits with an indeterminate form 0/0 can quite often be found by using algebraic simplification.

There are many more indeterminate forms other than **0/0**:

0^0, $1^{\#}$, # # #, #/#, 0 × #, and $\#^0$.

We will discuss these types later on in the course.

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Simplify and Try Again!

If it is not possible to substitute the value of x into the given equation of a function, try to simplify the expression in order to eliminate the zero in the denominator.

For Example:

1. Factor the denominator and the numerator, then try to cancel a zero (as seen in previous example).
2. If the expression consists of fractions, find a common denominator and then try to cancel out a zero (see example 3 on the next slides).
3. If the expression consists of radicals, rationalize the denominator by multiplying by the conjugate, then try to cancel a zero (see example 4 on the next slides).

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Examples:

$$3.\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h}=$$

$$4.\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=$$

Examples:

$$3.\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h}=$$

$$4.\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=$$

Answer

$$3.\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h}=\lim_{h\to 0}\frac{\frac{3-3-h}{(3+h)\cdot 3}}{h}=\lim_{h\to 0}\frac{\frac{-h}{(3+h)\cdot 3}}{h}=$$

$$\lim_{h\to 0}-\frac{1}{(3+h)\cdot 3}=-\frac{1}{9}$$

$$4.\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}=$$

$$\lim_{x\to 4}\frac{(x-4)\cdot(\sqrt{x}+2)}{x-4}=\lim_{x\to 4}(\sqrt{x}+2)=4$$

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Slide 100 (Answer) / 233

36 Find the limit: $\lim_{x \to -3} \frac{x+3}{x^2+5x+6} =$

- A $-\infty$
- B 0
- C -1
- D DNE
- E 1

36 Find the limit: $\lim_{x \to -3} \frac{x+3}{x^2+5x+6} =$

- A $-\infty$
- B 0
- C -1
- D DNE
- E 1

Answer

C

37 Find the limit: $\lim_{x \to 3} \frac{x^2+3x-18}{x^2-5x+6}$

- A ∞
- B 9
- C 0
- D $\frac{1}{9}$
- E $\frac{9}{5}$

37 Find the limit: $\lim_{x \to 3} \frac{x^2}{\cdots}$

Answer

B

- A ∞
- B 9
- C 0
- D $\frac{1}{9}$
- E $\frac{9}{5}$

Infinite Limits

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Infinite Limits

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Previously, we discussed the limits of rational functions with the indeterminate form 0/0.

Now we will consider rational functions where the denominator has a limit of zero, but the numerator does not. As a result, the function outgrows all positive or negative bounds.

Infinite Limits

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We can define a limit like this as having a value of *positive infinity* or *negative infinity*:

If the value of a function gets larger and larger without a bound, we say that the limit has a value of **positive infinity**.
If the value of a function gets smaller and smaller without a bound, we say that the limit has a value of **negative infinity**.

Next, we will use some familiar graphs to illustrate this situation.

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Infinite Limits

We can define a l... e
infinity or *negati*...

If the value of a f... ound,
we say that the l...
If the value of a fu... a
bound, we say th... **ity**.

Next, we will use so... situation.

Teacher Notes

Notice, that when the limit has a value of positive or negative infinity it still means that the limit does not exist. However, we can be more precise saying that a function is going to either positive or negative infinity.

Infinite Limits

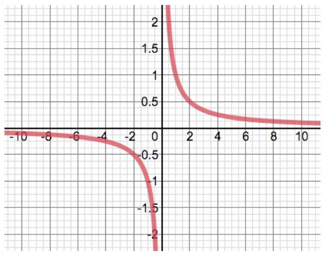
The figure to the right represents the function

$y = \frac{1}{x}$.

Can we compute the limit?:

$$\lim_{x \to 0} \frac{1}{x} = ?$$

$y = \frac{1}{x}$

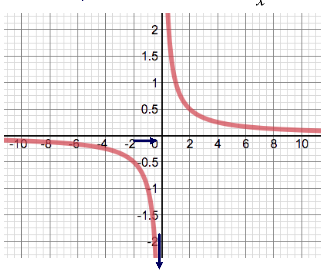


Left Hand Limit

In this case, we should first discuss one-sided limits. When x is approaching zero from the left the value of the function becomes smaller and smaller, so

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$y = \frac{1}{x}$

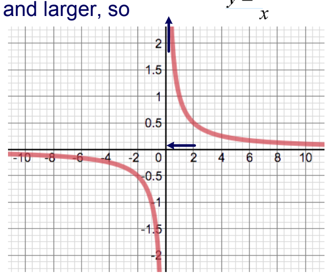


Right Hand Limit

When x is approaching zero from the right the value of function becomes larger and larger, so

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

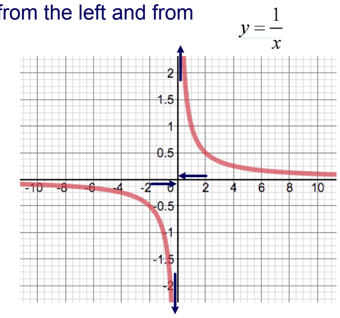
$y = \frac{1}{x}$



LHL ≠ RHL

By definition of a limit, a two-sided limit of this function does not exist, because the limit from the left and from the right are not the same:

$$\lim_{x \to 0} \frac{1}{x} = DNE$$

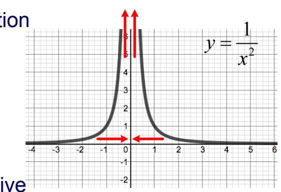


Infinite Limits

The figure on the right represents the function

$y = \frac{1}{x^2}$. *Can we compute the limit?:*

$$\lim_{x \to 0} \frac{1}{x^2} = ?$$

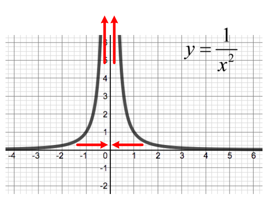


We see that the function outgrows all positive bounds as x approaches zero from the left and from the right, so we can say

$$\lim_{x \to 0^-} \frac{1}{x^2} = +\#, \quad \lim_{x \to 0^+} \frac{1}{x^2} = +\#$$

Thus, the two-sided limit is:

$$\lim_{x \to 0} \frac{1}{x^2} = +\#.$$



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Vertical Asymptote

As we recall from algebra, the vertical lines near which the function grows without bound are vertical asymptotes. Infinite limits give us an opportunity to state a proper definition of the vertical asymptote.

Definition

A line x = a is called **a vertical asymptote** for the graph of a function if either

$$\lim_{x \to a^-} f(x) = \pm\# \ , \quad \lim_{x \to a^+} f(x) = \pm\# \ ,$$

or

$$\lim_{x \to a} f(x) = \pm\# \ .$$

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Example

Find the vertical asymptote for the function $y = -\frac{1}{(x-3)^2}$.

First, let us sketch a graph of this function.

It is obvious from the graph, that

$\lim_{x \to 3} -\frac{1}{(x-3)^2} =$ _______.

So, the equation of the vertical asymptote for this function is _________.

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Example

Find the vertical asymptote fo

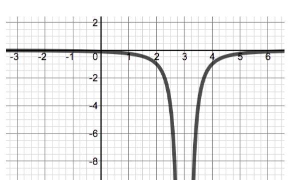
First, let us sketch a graph of

It is obvious from the graph t

$\lim_{x \to 3} -\frac{1}{(x-3)^2} =$ _______.

So, the equation of the vertic
for this function is _________

Answer



First blank: -#

Second blank: *x=3*

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Number Line Method

It seems that in the case when the denominator equals zero, but the numerator does not, as x approaches a certain number, we have to know what the graph looks like, before we can calculate a limit. Actually, this is not necessary.

There is **a number line method** that will help us to solve these types of problems.

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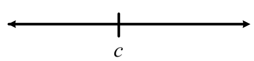
NUMBER LINE METHOD

If methods mentioned on previous pages are unsuccessful, you may need to use a Number Line to help you compute the limit. This is often helpful with one sided limits as well as limits involving absolute values, when you are not given a graph.

1. Make a number line marked with the value, "c" which x is approaching.



2. Plug in numbers to the right and left of "c"

Remember... if the limits from the right and left do not match, the overall limit DNE.

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Example 1

Find the limit: $\lim_{x \to 3} \frac{5-x}{(x-3)(x+1)} = ?$

Step 1. Find all values of x that are zeros of the numerator and the denominator:

Step 2. Draw a number line, plot these points.

Example 1

Find the limit: lir

Step 1. Find all v ... tor and the denominator

Step 2. Draw a r

Answer

Step 1: x=-1, x=3, x=5

Step 2:

-1 3 5

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Example 1

Step 3. Using the number line, test the sign of the value of thefunction at numbers inside the zeros-interval, near 3, on both the left and right sides. (for example, pick *x=2* and *x=4).*

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Example 1

Step 3. Us ... n at numbe ... sides. (fo

Answer

-1 -------------- 3 ++++ 5

$f(2)=\frac{5-x}{(x-3)(x+1)}=-1,$ that means $\lim_{x\to3^-}\frac{5-x}{(x-3)(x+1)}=-\infty$

$f(4)=\frac{5-x}{(x-3)(x+1)}=\frac{1}{5},$ that means $\lim_{x\to3^+}\frac{5-x}{(x-3)(x+1)}=+\infty$

Finally, $\boxed{\lim_{x\to3}\frac{5-x}{(x-3)(x+1)}=DNE}$

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Example 2

Use the number line method to find the limit:

$$\lim_{x\to 6}\frac{2-x}{(x-6)^2(x-4)}=?$$

Example 2

Use the number line method to

$$\lim_{x\to 6}\frac{}{(x-}$$

Answer

-#

40 Find the limit: $\lim_{x\to 11^-}\frac{x+2}{x-11}=$ ___

41 Find the limit $\lim_{x \to 11^+} \frac{x+2}{x-11} =$ ___

41 Find the limit $\lim_{x \to 11^+} \frac{x+2}{x-11}$

Answer

try $x = 12$

$f(12) = \frac{14}{1}$

that means, $\lim_{x \to 11^+} = +\infty$

Answer: **+#**

42 Find the limit: $\lim_{x\to 11} \frac{x+2}{x-11} = \_\_\_$

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42 Find the limit: $\lim_{x\to 11} \frac{x+}{x-}$

Answer

From the previous two problems, one leads to $-\infty$ the other leads to $+\infty$

Answer: **DNE**

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Limits of Absolute Value and Piecewise-Defined Functions

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Abs. Value & Piecewise Limits

In the beginning of the unit we used the graphical approach to obtain the limits of the absolute value and the piecewise-defined functions. However, we do not have to graph the given functions every time we want to compute a limit.

Now we will offer algebraic methods to find limits of those functions. There is a reason why we discuss the absolute value and the piecewise functions in the same section: the graphs of these functions have two or more parts that are given by different equations. When you are trying to calculate a limit you have to be clear of which equation you have to use.

Limits involving Absolute Values

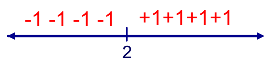
While the previous examples were very straight forward, how do we approach the following situation?:

$$\lim_{x\to 2}\frac{|x-2|}{x-2}=?$$

It is not possible here to substitute the value of x into the given formula. But we can calculate one-sided limits from the left and from the right. Any number that is bigger than 2 will turn the given expression into 1 and any number that is less than 2 will turn this expression into -1, so:

$$\lim_{x\to 2^+}\frac{|x-2|}{x-2}=1$$

$$\lim_{x\to 2^-}\frac{|x-2|}{x-2}=-1$$



Therefore, $\lim_{x\to 2}\frac{|x-2|}{x-2}=DNE$

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Example:

Find the indicated limits:

1. $\lim_{x\to -5^-}\frac{|x+5|}{x+5}=$ ______

2. $\lim_{x\to -5^+}\frac{|x+5|}{x+5}=$ ______

3. $\lim_{x\to -5}\frac{|x+5|}{x+5}=$ ______

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Example:

Find the indicated limits:

1. $\lim_{x\to -5^-}\frac{|x+5|}{x+5}=$ ______

2. $\lim_{x\to -5^+}\frac{|x+5|}{x+5}=$ ______

3. $\lim_{x\to -5}\frac{|x+5|}{x+5}=$ ______

Answer

1. $\lim_{x\to -5^-}\frac{|x+5|}{x+5}=-1$

2. $\lim_{x\to -5^+}\frac{|x+5|}{x+5}=1$

3. $\lim_{x\to -5}\frac{|x+5|}{x+5}=DNE$

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46 Find the limit:

$$\lim_{x\to 0^+} \frac{|x|}{x}$$

46 Find the limit:

$$\lim_{x\to 0^+} \frac{|x|}{x}$$

Answer

1

47 Find the limit:

$$\lim_{x\to 0^-} \frac{|x|}{x}$$

47 Find the limit:

$$\lim_{x\to 0^-}\frac{|x|}{x}$$

Answer

-1

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48 Find the limit:

$$\lim_{x\to 0}\frac{|x|}{x}$$

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48 Find the limit:

$$\lim_{x\to 0}\frac{|x|}{x}$$

Answer

DNE

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49 Find the limit:

$$\lim_{x\to 0^-} \frac{5|x|}{x}$$

49 Find the limit:

$$\lim_{x\to 0^-} \frac{5|x|}{x}$$

Answer

-5

50 Find the limit:

$$\lim_{x\to 0^+} \frac{5|x|}{x}$$

50 Find the limit:

$$\lim_{x \to 0^+} \frac{5|x|}{x}$$

Answer

5

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51 Find the limit:

$$\lim_{x \to 0} \frac{5|x|}{x}$$

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51 Find the limit:

$$\lim_{x \to 0} \frac{5|x|}{x}$$

Answer

DNE

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Piecewise Limits

The key to calculating the limit of a piecewise function is to identify the interval that the x value belongs to. We can simply compute a limit of the function that is represented by the equation when x lies inside that interval. It may sound a lit bit tricky, however it is quite simple if you look at the next example.

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Two-Sided Piecewise Limits

A two-sided limit of the piecewise function at the point where the formula changes is best obtained by first finding the one sided limits at this point.

If limits from the left and the right equal the same number, the two-sided limit exists, and *is* this number. If limits from the left and the right are not equal, than the two-sided limit does not exist.

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Example:

Find the indicated limit of the piecewise function:

$$\lim_{x \to -3} f(x) = ? \qquad f(x) = \begin{cases} \frac{1}{x+3}; x < -3 \\ x^2 - 12; -3 < x \le 4 \\ \sqrt{x+12}; x > 4 \end{cases}$$

1. First we will calculate the one-sided limit from the left of the function represented by which formula?

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Example:

Find the indicated limit of the piecewise function:

$$\lim_{x \to 4} f(x) = ? \qquad f(x) = \begin{cases} \dfrac{1}{x+3}; x < -3 \\ x^2 - 12; -3 < x \le 4 \\ \sqrt{x+12}; x > 4 \end{cases}$$

1. First, we will calculate one-sided limit from the left of the function represented by which formula?

Example:

Find the indicated limit of the pie

$\lim_{x \to 4} f(x) = ?$ $f(x)$

Answer

Calculate the limit from the left using this formula.

$f(x) = x^2 - 12$

$\lim_{x \to 4^-} f(x)$

$\lim_{x \to 4^-} (x^2 - 12) = 4$

1. First, we will calculate one-s ion represented by which formula?

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Example:

Find the indicated limit of the piecewise function:

$$\lim_{x \to 4} f(x) = ? \qquad f(x) = \begin{cases} \frac{1}{x+3}; x < -3 \\ x^2 - 12; -3 < x \le 4 \\ \sqrt{x+12}; x > 4 \end{cases}$$

2. Then we can calculate one-sided limit from the right of the function represented by which formula?

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Limits of End Behavior

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End Behavior

In the previous sections we learned about an indeterminate form 0/0 and vertical asymptotes. The indeterminate forms such as 1^{∞}, $\infty - \infty$, ∞/∞, 1/# and others will lead us to the discussion of the end of the behavior function and the horizontal asymptotes.

Definition

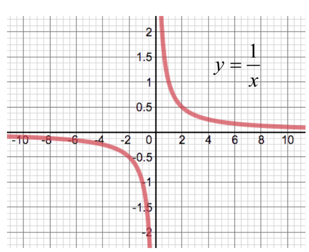
The behavior of a function f(x) as x increases or decreases without bound (we write $x \to +\infty$ or $x \to$ -#) is called **the end behavior of the function.**

End Behavior

Let us recall the familiar function $y = \frac{1}{x}$

Can we compute the limits:

$$\lim_{x \to -\infty} \frac{1}{x} = ? \qquad \lim_{x \to +\infty} \frac{1}{x} = ?$$



Let us recall the familiar

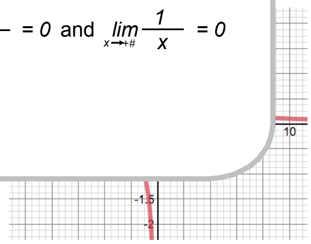
Can we compute the lim

$$\lim_{x\to-\infty}\frac{1}{x} = ?$$

Answer

From the graphical approach it is clear:

$$\lim_{x\to-\infty}\frac{1}{x} = 0 \text{ and } \lim_{x\to+\infty}\frac{1}{x} = 0$$



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End Behavior and Horizontal Asymptotes

In general, we can use the following notation.

If the value of a function *f(x)* eventually get as close as possible to a number *L* as *x* increases without bound, then we write:

$$\lim_{x\to+\infty} f(x) = L$$

Similarly, if the value of a function f(x) eventually get as close as possible to a number *L* as x decreases without bound, then we write:

$$\lim_{x\to-\infty} f(x) = L$$

We call these limits **Limits at Infinity** and the line ***y=L*** is **the horizontal asymptote of the function f(x).**

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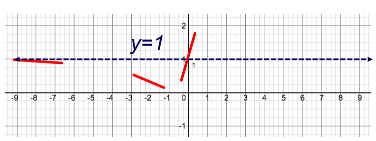
End Behavior

The figure below illustrates the end behavior of a function and the horizontal asymptotes ***y=1.***

$$\lim_{x\to+\infty} f(x) = 1$$

$$\lim_{x\to-\infty} f(x) = 1$$



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Infinite Limits

Consider $\lim_{x\to+\infty}\frac{x^3}{x^2}=\frac{+\infty}{+\infty}$ Is the limit 1?

It is not, because if we reduce the rational expression before substituting, we will get the limit :

$$\lim_{x\to+\infty} x = +\infty$$

In calculus, we have to divide each term by the highest power of x, then take the limit. Remember that a number/# is 0 as we have seen in a beginning of the section. For example,

$$\lim_{x\to+\infty}\frac{2}{x}=0$$

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Rule 2

Rule 2. If the highest power of *x* appears in the numerator (top heavy), then

$$\lim_{x\to\pm\infty} f(x) = \pm\infty$$

In order to determine the resulting sign of the infinity you will need

to plug in very large positive or very large negative numbers . Look closely at the examples on next pages to understand this rule clearly.

Examples

$$1a.\ \lim_{x\to+\infty} \frac{5x^7 - 2x^5 + 15}{2x^4 + 21x^3 + 3x} = +\infty$$

For rational functions, the end behavior matches the end behavior of the quotient of the highest degree term in the numerator divided by the highest degree term in the denominator. So, in this case the function will behave as $y=x^3$. When *x* approaches positive infinity, the limit of the function will go to positive infinity.

$$1b.\ \lim_{x\to-\infty} \frac{5x^7 - 2x^5 + 15}{2x^4 + 21x^3 + 3x} = -\infty$$

When *x* approaches negative infinity in the same problem, the limit of function will go to negative infinity.

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Try These on Your Own

$$\lim_{x\to\infty}\frac{4x+5}{3x-1} = \_\_\_$$

$$\lim_{x\to-\infty}\frac{5x^2+3x-1}{4x+7} = \_\_\_$$

$$\lim_{x\to\infty}\frac{5x^2+7}{x^3+1} = \_\_\_$$

$$\lim_{x\to-\infty}\frac{2x^5-11x^7}{6x^2+21} = \_\_\_$$

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Try These on Your Own

Answer

$$\lim_{x\to\infty}\frac{4x+5}{3x-1} = \frac{4}{3}$$ rule 3

$$\lim_{x\to-\infty}\frac{5x^2+3x-1}{4x+7} = -\infty$$ rule 2, example 1b

$$\lim_{x\to\infty}\frac{5x^2+7}{x^3+1} = 0$$ rule 1

$$\lim_{x\to-\infty}\frac{2x^5-11x^7}{6x^2+21} = +\infty$$ rule 2

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59 Find the indicated limit.

$$\lim_{x \to +\infty} \frac{2}{x-4} =$$

59 Find the indicated limi

$$\lim_{x \to +\infty} \frac{2}{x-4} =$$

Answer

0

61 $\lim_{x \to -\infty} \frac{3x^3 + 4x - 2}{4x^3 + 5x} =$

- A 0
- B $\frac{3}{4}$
- C $\frac{-3}{4}$
- D $-\infty$

61 $\lim_{x \to -\infty} \frac{3x^3+4x-2}{4x^3+5x} =$

Answer

B

A 0

B $\frac{3}{4}$

C $\frac{-3}{4}$

D $-\infty$

62 $\lim_{x \to -\infty} \frac{3x^3+4x-2}{\sqrt{4x^6-7}+5x}$

A 0

B $\frac{3}{2}$

C $\frac{-3}{2}$

D $-\infty$

62 $\lim_{x \to -\infty} \frac{3x^3+4x-2}{\sqrt{4x^6-7}+5}$

Answer

C

A 0

B $\frac{3}{2}$

C $\frac{-3}{2}$

D $-\infty$

Using Conjugates

Use conjugates to rewrite the expression as a fraction,then solve like # /#.

$$\lim_{x\to\infty}\left(x^3 - x^2\right)$$

$$\lim_{x\to\infty}\left(x^3 - x^2\right)\left(\frac{x^3 + x^2}{x^3 + x^2}\right)$$

$$\lim_{x\to\infty}\left(\frac{x^6 - x^4}{x^3 + x^2}\right)$$

$$\lim_{x\to\infty}\left(\frac{\frac{x^6}{x^6} - \frac{x^4}{x^6}}{\frac{x^3}{x^6} + \frac{x^2}{x^6}}\right) = \lim_{x\to\infty}\left(\frac{1 - \frac{1}{x^2}}{\frac{1}{x^3} + \frac{1}{x^4}}\right) = \frac{1-0}{0+0} = \frac{1}{0} = +\infty$$

Example

Evaluate: $\lim_{x\to\infty}\left(\sqrt{x^8 + 5} - x^4\right)$

Evaluate: $\lim_{x\to\infty}\left(\sqrt{\right.$

Answer

$$=\lim_{x\to\infty}\left(\sqrt{x^8+5}-x^4\right)\left(\frac{\sqrt{x^8+5}+x^4}{\sqrt{x^8+5}+x^4}\right)$$

$$=\lim_{x\to\infty}\left(\frac{x^8+5-x^8}{\sqrt{x^8+5}+x^4}\right)$$

$$=\lim_{x\to\infty}\left(\frac{5}{\sqrt{x^8+5}+x^4}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\frac{5}{x^4}}{\sqrt{\frac{x^8}{x^8}+\frac{5}{x^8}}+\frac{x^4}{x^4}}\right)=\lim_{x\to\infty}\left(\frac{\frac{5}{x^4}}{\sqrt{1+\frac{5}{x^8}}+1}\right)=\frac{0}{\sqrt{1+0}+1}=0$$

63 $\lim_{x\to\infty}\left(\sqrt{x^6+5x^3}-x^3\right)$

63 $\lim_{x\to\infty}\left(\sqrt{x^6+5x^3}-\right.$

Answer

$$\frac{5}{2}$$

Trig Limits

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Examples

$$1. \lim_{x \to 0} \frac{\sin 4x}{4x} =$$

$$2. \lim_{x \to 0} \frac{\sin 4x}{2x} =$$

$$3. \lim_{x \to 0} \frac{\sin 5x}{\sin 3x} =$$

Answer

1. 1

2. $\lim_{x\to 0}\frac{2\sin 4x}{2\cdot 2x}=2(1)$

3. $\lim_{x\to 0}\frac{\frac{\sin 5x}{x}}{\frac{\sin 3x}{x}}=\lim_{x\to 0}\frac{\frac{5\sin 5x}{5x}}{\frac{3\sin 3x}{3x}}=\frac{5\cdot 1}{3\cdot 1}=\frac{5}{3}$

$3.\lim_{x\to 0}\frac{\sin 5x}{\sin 3x}=$

64 $\lim_{x\to 0}\frac{\sin 5x}{x} = ?$

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64 $\lim_{x\to 0}$

Answer

5

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65 $\lim_{x\to 0}\frac{2x}{\sin 3x} = ?$

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65 $\lim_{x\to 0}$

Answer

2/3

66 $\lim_{x\to 0}\frac{x}{\cos x-1}=?$

66 $\lim_{x\to 0}$

Answer

DNE

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67 $\lim_{x\to 0}\frac{1-\cos^2 4x}{\sin^2 3x}=?$

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67 $\lim_{x\to 0}\frac{1}{}$

Answer

16/9

Solution:

$$\lim_{x\to 0}\frac{1-\cos^2 4x}{\sin^2 3x}=\lim_{x\to 0}\frac{\sin^2 4x}{\sin^2 3x}=$$

$$\lim_{x\to 0}\frac{\sin 4x\cdot\sin 4x}{\sin 3x\cdot\sin\ 3x}=$$

$$\lim_{x\to 0}\frac{\frac{\sin 4x\cdot\sin 4x}{16x^2}\cdot 16x^2}{\frac{\sin 3x\cdot\sin 3x}{9x^2}\cdot 9x^2}=\frac{16}{9}$$

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68 $\lim_{x\to 0}\frac{\tan^2 x}{3x}=?$

68 $\lim_{x\to 0}$

Answer

0

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Limits Summary & Plan of Attack

It can seem very overwhelming to think about all the possible strategies used for limit questions. Hopefully, the next pages will provide you with a game plan to approach limit problems with confidence.

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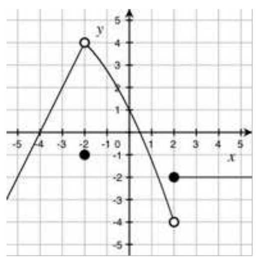
Limits with Graphs



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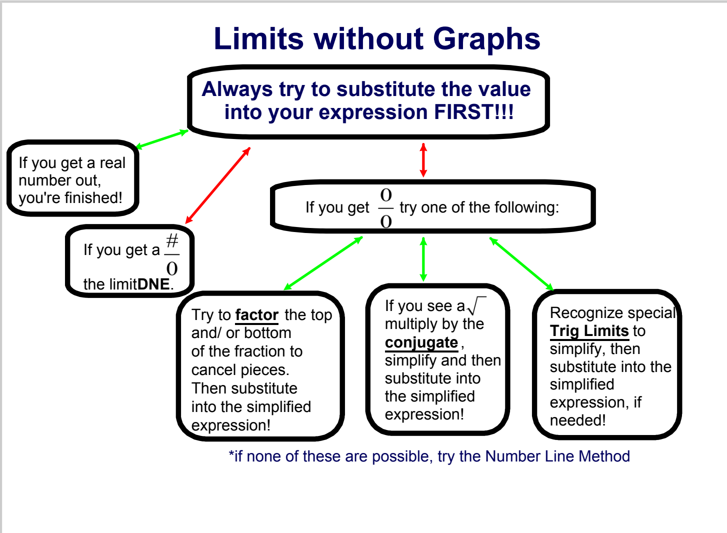
Teacher Notes

Use this slide to emphasize the 5 different types of questions your students may be asked.

Use the given graph to ask students different questions at specific x- values.

For example:

- What is f(-2)?
- What is the lim $_{x\to -2^+}$?
- What is the lim $_{x\to -2^-}$?
- What is the lim $_{x\to -2}$?
- Is the graph continuous at -2?



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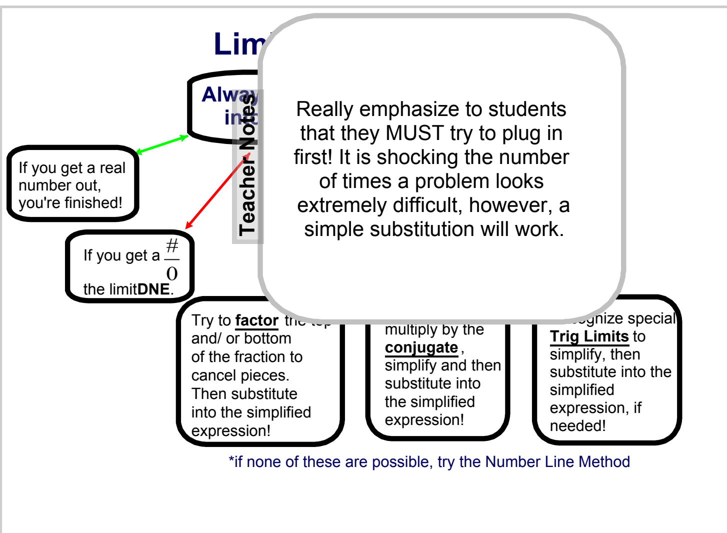

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Continuity

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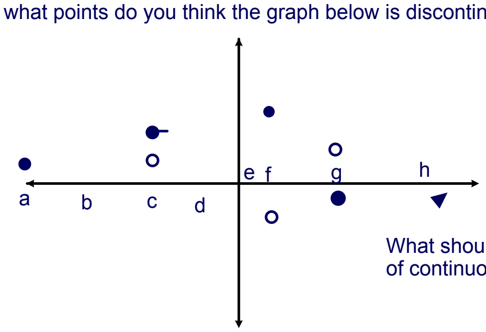
Continuity

At what points do you think the graph below is continuous?
At what points do you think the graph below is discontinuous?



What should the definition of continuous be?

AP Calculus Definition of Continuous

1) f(a) exists

2) $\lim_{x \to a} f(x)$ exists

3) $\lim_{x \to a} f(x) = f(a)$

This definition shows continuity at a point on the interior of a function.

For a function to be continuous, every point in its domain must be continuous.

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Continuity at an Endpoint

Replace step 3 in the previous definition with:

Left Endpoint: $\lim_{x \to a^+} f(x) = f(a)$

Right Endpoint: $\lim_{x \to a^-} f(x) = f(a)$

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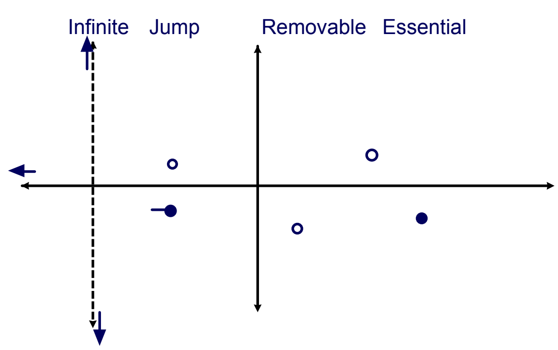
Types of Discontinuity



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70 Given the function decide if it is continuous or not. If it is not state the reason it is not.

$$y = |x|$$

- A continuous
- B
- C $\lim_{x \to a} f(x)$ *does not exist for all a*
- D $\lim_{x \to a} f(x) = f(a)$ *is not true for all a*

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70 Given the function decide if it is continuous or not. If it is not state the reason it is not.

$$y=|$$

A continuous

B

C $\lim_{x\to a} f(x)$ *does not*

D $\lim_{x\to a} f(x)=f(a)$ *is n*

Answer

A

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71 Given the function decide if it is continuous or not. If it is not state the reason it is not.

$$y=\frac{1}{x}$$

A continuous

B *a*

C *f(a) does not exist for all a and* $\lim_{x\to a} f(x)$ *does not exis*

D $\lim_{x\to a} f(x)=f(a)$ *is not true for all a*

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71 Given the function decide if it is continuous or not. If it is not state the rea

$$y=$$

A continuous

B

C *f(a) does not exist* *exis*

D $\lim_{x\to a} f(x)=f(a)$

Answer

C

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74 What value(s) would remove the discontinuity(s) of the given function?

$$f(x)=\frac{x^2+3x}{x+3}$$

- ❑ A -3
- ❑ B -2
- ❑ C -1
- ❑ D -1/2
- ❑ E 0
- ❑ F 1/2
- ❑ G 1
- ❑ H 2
- ❑ I 3
- ❑ J DNE

74 What value(s) would remove the discontinuity(s) of the given function?

$$f(x)=$$

Answer

A: $f(-3)=-3$

- ❑ A -3
- ❑ B -2
- ❑ C -1
- ❑ D -1/2
- ❑ E 0
- ❑ F
- ❑ G
- ❑ H 2
- ❑ I 3
- ❑ J DNE

75 What value(s) would remove the discontinuity(s) of the given function?

$$f(x)=\frac{\sin x}{2x}$$

- ❑ A -3
- ❑ B -2
- ❑ C -1
- ❑ D -1/2
- ❑ E 0
- ❑ F 1/2
- ❑ G 1
- ❑ H 2
- ❑ I 3
- ❑ J DNE

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75 What value(s) would remove the discontinuity(s) of the given function?

$f(x)$

Answer

E: $f(0)=1/2$

- A -3
- B -2
- C -1
- D -1/2
- E 0
- F
- G
- H
- I 3
- J DNE

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76 What value(s) would remove the discontinuity(s) of the given function?

$$f(x)=\frac{x-2}{|x|-2}$$

- A -3
- B -2
- C -1
- D -1/2
- E 0
- F 1/2
- G 1
- H 2
- I 3
- J DNE

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76 What value(s) would remove the discontinuity(s) of the given function?

$f(x)$

Answer

B: $f(-2)=DNE$

H: $f(2)=1$

- A -3
- B -2
- C -1
- D -1/2
- E 0
- F 1/2
- G 1
- H 2
- I 3
- J DNE

Making a Function Continuous

$f(x)=\begin{cases} x^2-1; x<3 \\ 2ax; x\geq 3 \end{cases}$ *, find a so that f(x) is continuous.*

Both 'halves' of the function are continuous. The concern is making $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)$

Making a Function Continuous

$f(x)=\begin{cases} x^2-1; x<3 \\ 2ax; x\geq 3 \end{cases}$ *, find* ... *s.*

Both 'halves' of the function are ... making $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)$

Answer

$$3^2-1=2a(3)$$
$$8=6a$$
$$\frac{4}{3}=a$$

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Intermediate Value Theorem

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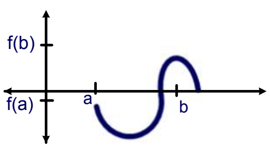
The Intermediate Value Theorem

The characteristics of a function on closed continuous interval is called The Intermediate Value Theorem.

If f(x) is a continuous function on a closed interval [a,b], then f(x) takes on every value between f(a) and f(b).

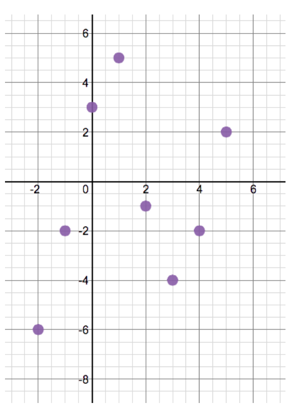


This comes in handy when looking for zeros.

Finding Zeros

Can you use the Intermediate Value Theoremto find the zeros of this function?

X	Y
-2	-6
-1	-2
0	3
1	5
2	-1
3	-4
4	-2
5	2



Answer

79 Give the letter that lies inthe same interval as a zero of this continuous function.

A B C D

X	1	2	3	4	5
Y	-2	-1	0.5	2	3

Answer

80 Give the letter that lies in the sameinterval as a zero of this continuous function.

A B C D

X	1	2	3	4	5
Y	4	3	2	1	-1

Answer

81 Give the letter that lies in thesame interval as a zero of this continuous function.

A B C D

X	1	2	3	4	5
Y	-2	-1	-0.5	2	3

Answer

Difference Quotient

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Difference Quotient

Now that you are familiar with the different types of limits, we can discuss real life applications of this very important mathematical term. You definitely noticed that in all formulas stated in the previous section, the numerator is presented as a difference of a function or, in another words, as a change of a function; and the denominator is a point difference. Such as:

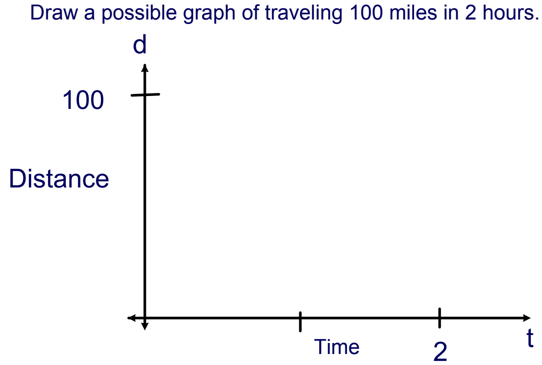
$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{\Delta x} = \frac{\Delta f(x)}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{h}$$

We call this a Difference Quotient or an Average Rate of Change.

If we consider a situation when x_1 approaches x_0 ($x_1 \to x_0$), which means $\Delta x \to 0$, or $h \to 0$

Example

Draw a possible graph of traveling 100 miles in 2 hours.



Average Rate of Change

Using the graph on the previous slide:
What is the average rate of change for the trip?

Is this constant for the entire trip?

What formula could be used to find the average rate of change between 45 minutes and 1 hour?

Average Rate of Change

The slope formula of $\frac{d_2 - d_1}{t_2 - t_1}$ represents the Velocity or Average Rate of Change. This is the slope of the secant line from (t_1,d_1) to (t_2,d_2).

Suppose we were looking for Instantaneous Velocity at 45 minutes, what values of (t_1,d_1) and (t_2,d_2) should be used?

Is there a better approximation?

The Difference Quotient

The closer (t_1,d_1) and (t_2,d_2) get to one another the better the approximation is.

Let h represent a very small value so that (x, f(x)) and (x+h, f(x+h)) are 2 points that are very close to each other.

The slope between them would be $\frac{f(x+h)-f(x)}{(x+h)-x}$

And since we want h to "disappear" we use

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{(x+h)-x} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

This is called the Difference Quotient.

Derivative

The Difference Quotient gives the instantaneous velocity, which is the slope of the tangent line at a point.

A derivative is used to find the slope of a tangent line. So, the Difference Quotient can be used to find a derivative algebraically.

Example of the Difference Quotient

Find the slope of the tangent line to the function $f(x) = 2x^2 - x + 4$ at x=3.

Exa

Find the s
at x=3.

Answer

$$\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$$

$$= \lim_{h\to 0}\frac{\left(2(3+h)^2-(3+h)+4\right)-\left(2(3)^2-(3)+4\right)}{h}$$

$$= \lim_{h\to 0}\frac{\left(2(9+6h+h^2)-(3+h)+4\right)-\left(2(9)-(3)+4\right)}{h}$$

$$= \lim_{h\to 0}\frac{\left(18+12h+2h^2-3-h+4\right)-(19)}{h}$$

$$= \lim_{h\to 0}\frac{\left(2h^2+11h+19\right)-(19)}{h} = \lim_{h\to 0}\frac{2h^2+11h}{h} = \lim_{h\to 0}2h+11 = 11$$

Example of the Difference Quotient

Find an equation that can be used to find the slope of the tangent line at any point on the function $f(x) = 3x^3 - 8$

Example of the Difference Quotient

Find an equ... line at any point

Answer

$$\lim_{h\to 0}\frac{\left(3(x+h)^3-8\right)-\left(3x^3-8\right)}{h}$$

$$=\lim_{h\to 0}\frac{\left(3(x^3+3x^2h+3xh^2+h^3)-8\right)-\left(3x^3-8\right)}{h}$$

$$=\lim_{h\to 0}\frac{3x^3+9x^2h+9xh^2+3h^3-8-3x^3+8}{h}$$

$$=\lim_{h\to 0}\frac{9x^2h+9xh^2+3h^3}{h}$$

$$=\lim_{h\to 0}\left(9x^2+9xh+3h^2\right)=9x^2$$

To find the slope atx=2, sub 2 into $9x^2$.
So the slope at x=2 is 36.

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